On operator algebras, linear logic and categorical quantum mechanics

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Introduction

- This presentation is about work in progress and preliminary ideas.
- Therefore, mainly ideas and conjectures and not really any results.
- Please discuss with me if any of these ideas interest you!

What are operator algebras?

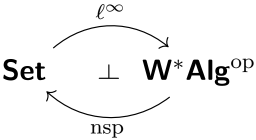
- Certain kinds of normed rings (precisely, associative algebras) over $\mathbb{R}$ or $\mathbb{C}$
- Of special interest are C^*- and von Neumann (W^*-) algebras
- Canonical example: the algebra of bounded linear operators on an Hilbert space forms a C^*- (in fact, W^*-)algebra
- Form categories $\mathbf{C^*Alg}$ and $\mathbf{W^*Alg}$
- We focus on W^*-algebras here, and assume all algebras are unital (but not necessarily commutative)

Categorical properties of $\mathbf{W^*Alg}$

- Theorem [Kornell 2016]: $\mathbf{W^*Alg}$ is symmetric monoidal with the *spatial tensor product*
- Model of linear logic? Even better!
- Theorem [Kornell 2016]: $\mathbf{W^*Alg}^{\mathrm{op}}$ is monoidal closed with the *free exponential construction*!
- Intuition: $\mathbf{W^*Alg}^{\mathrm{op}}$ behaves like a "quantum version" of **Set**

Operator algebra models of linear logic

- Naturally, we ask if operator algebras can form a model of linear logic, like $\mathbf{Vect}_K$ or **FHilb**
- Answer: yes!
- Theorem [Cho & Westerbaan 2016A]: **Set** and $\mathbf{W^*Alg}^{\mathrm{op}}$ form a *linear-non-linear* model of linear logic in the sense of [Benton 1994]



Small ideas & conjectures

- $\mathbf{W^*Alg}^{\mathrm{op}}$ is actually a *Lafont category*! [Cho & Westerbaan 2016B] Unique duplication?
- Minor idea: verify the explicit constrution for free commutative comonoid [Melliès et al. 2009]
- What about *classical* linear logic? [Paykin & Zdancewic 2015] gives a Benton-style categorical construction of models of CLL; is $\mathbf{W^*Alg}^{\mathrm{op}}$ a model?
- Conjecture: $\mathbf{W^*Alg}^{\mathrm{op}}$ gives a model of classical linear logic

Other structures on $\mathbf{W^*Alg}^{\mathrm{op}}$

- *Geometry of Interaction* (GoI) has a close connection to quantum operations in QM
- Conjecture: a finite-dimensional subcategory of $\mathbf{W^*Alg}^{\mathrm{op}}$ is traced monoidal ($\approx$ model of GoI)
- Naturally, one may want to extend it to *differential linear logic*
- Conjecture: $\mathbf{W^*Alg}^{\mathrm{op}}$ is a *differential category* ([Blute et al. 2006]), making it a model of DLL
- The finite-dimensional case is likely uninteresting ([Lemay 2019]), but the all-dimension case is very hopeful

Can we replace **Set** and $\mathbf{W^*Alg}^{\mathrm{op}}$?

- What about the full category $\mathbf{C^*Alg}^{\mathrm{op}}$?
- Theorem [Gelfand]: $\mathbf{C^*Alg}_{\mathrm{com}} \cong \mathbf{CHaus}^{\mathrm{op}}$! This is the famous *Gelfand duality*, what are its implications here?
- Theorem [Pavlov 2021]: $\mathbf{W^*Alg}^{\mathrm{op}}_{\mathrm{com}}$ is equivalent to the category of compact strictly localizable enhanced measurable spaces; connections to probability?
- Another category: f.d. Hilbert spaces and quantum relations of [Weaver 2010]
- This category embeds into $\mathbf{W^*Alg}^{\mathrm{op}}$; [Kornell 2021] calls this the category of *quantum sets*

Non-commutativity, quantum physics and logic

Geometry	CT	Logic	Physics
commutative	cartesian	non-linear	classical
non-commutative	sym. mon.	linear	quantum
(new geometry)	braided	braided	TQFT

This is just an intuition, nothing serious happening yet! But what can we do with this big picture?